Chapter 10

The paratonal equation

10.1 Background

Up until now, various topics from a range of fields have been introduced that are deemed critical for the understanding of the temporal imaging theory, which we shall start presenting in this chapter. The concepts that were presented earlier will be used throughout this work—some more than others—and to a large degree serve to motivate it.

The basis for the temporal imaging theory is a solution to the scalar Helmholtz wave equation that was developed in the late 1960s, but has never been used in acoustics. There are two features of this solution that make it particularly attractive in hearing. First, it deals with the complex envelope of the signal and not with the carrier, which is immediately useful in problems of modulation, where nonstationarity is key. It therefore dodges the static nature of classical Fourier analysis that has us infer modulation indirectly from its spectrum, or apply time-windows on otherwise infinitely long transforms (Blinchikoff and Zverev, 2001, pp. 383–395). Second, it is mathematically analogous to the paraxial equation for light. This means that it should be possible to devise an analogous imaging system in sound by combining dispersive elements, in analogy to spatial optics, where diffraction is used instead. Both these aspects have far-reaching implications to hearing theory—some of which will be explored in the final chapters of this work.

The temporal paraxial equation was derived by the nonlinear-optics physicist Sergeĭ Aleksandrovich Akhmanov and colleagues in Akhmanov et al. (1968, 1969)[93]. It was done in the context of a comprehensive space-time equivalence theory, which exploits mathematical analogies between time-modulated to narrow bounded-beam waves to account for the propagation of **ultrashort pulses** of high-energy laser in dispersive media. In fact, using different considerations, a completely independent treatment of dispersive waves got close to a complete optical imaging system analogy was presented earlier by Pierre Tournois (1964). He also proposed to apply this solution to acoustic waves (Tournois, 1967), but that work has never been followed up. Related concepts have been applied even earlier, in chirp radar technology, using a technique called **pulse compression** (Klauder et al., 1960). Pulse compression in radars employs frequency modulation (FM) to obtain high-power signals that have a large time-bandwidth product. It requires matched filtering at the receiver, which inverts the FM and undoes the pulse compression. In applied optics, dispersion has been successfully employed to stretch, amplify, and recompress an ultrashort low-energy laser pulse into a powerful one (Strickland and Mourou, 1985)[94]. While we adhere to the formalism developed in optics, it should be noted that ideas from radar technology have been influential in the context of bat echolocation for a long time (Griffin, 1958, pp. 342–346), including pulse compression signal processing.

[93]See Drabovich and Chirkin, 1999 for a short professional biography of Akhmanov.

[94]This seminal experiment won its authors a Nobel Prize in Physics in 2018.

The basis for the space-time analogy is valid for carrier waves of visible light, such as those carried by thin, axially uniform, single-mode optic fibers, which can be modulated by low-frequency microwaves, in a source-free space (Haus, 1984, pp. 178–187). The description of this propagation is scalar, because electromagnetic polarization that would have made the description vectorial, may be neglected. It means that with careful inspection of the assumptions, it is straightforward to adapt the plane-wave scalar equations of light to the plane waves of sound. Therefore, in the following we will derive the "paraxial" dispersion equation and follow with a presentation of the time lens.

10.2 The "paraxial" approximation of the dispersion equation

We consider the propagation of a plane wave at $z \geq 0$ with known spectrum at $z = 0$. This refers to a so-called secondary source, according to Huygens principle, which treats any point on the wavefront away from the source as a point source (§4.2.2).

Starting from the homogeneous three dimensional acoustic wave equation (e.g., Morse and Ingard, 1968, p. 282),

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial^2 t} \tag{10.1}$$

Where p is the pressure, and c is the speed of sound. In the plane-wave approximation, the direction of propagation is arbitrarily set parallel to z, so that the wavenumber components $k_x = k_y = 0$ and the profile of the propagating wave may be ignored (i.e., an infinite plane wavefront). This results in a one-dimensional equation

$$\frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial^2 t} \tag{10.2}$$

A monochromatic solution that satisfies Eq. 10.2 can then take the general form (§3.2.1)

$$p(x, y, z, t) = p_0 e^{i[\omega_c t - k(\omega)z]} \tag{10.3}$$

Where p_0 is the pressure wave amplitude, and $k(\omega)$ is the frequency-dependent wavenumber in the medium, which may entail dispersion, if $k(\omega) \neq \text{const.}$

The derivation of the dispersion equation below follows Haus (1984, pp. 179–181) and New (2011, pp. 128–129) for scalar electromagnetic fields. The central assumption in this derivation is that the pressure envelope varies slowly in space, so that the modulation wavelength is much larger than the carrier, $\lambda_m \gg \lambda_c$, and the corresponding period $T_m \gg T_c$ (Akhmanov et al., 1968).

Let us posit a known frequency dependence of the medium dispersion $k(\omega)$ and a corresponding dependence on z of the spectrum $P(z, \omega)$, assuming that $p(z, t)$ is square-integrable, so it has a Fourier representation

$$P(z, \omega) = A(\omega, 0)e^{-ik(\omega)z} \tag{10.4}$$

Where $A(\omega, 0)$ is the initial complex spectral envelope of $P(z, \omega)$. Then, along the z-axis, the following differential equation is satisfied

$$\frac{\partial P(z, \omega)}{\partial z} = -ik(\omega)P(z, \omega) \tag{10.5}$$

For a slowly-varying complex envelope, the dispersion relation of $k(\omega)$ can be approximated using Taylor series around ω_c

$$k(\omega) = k_c + \frac{dk}{d\omega}(\omega - \omega_c) + \frac{1}{2}\frac{d^2k}{d\omega^2}(\omega - \omega_c)^2 + ... \tag{10.6}$$

$k(\omega)$ is a complex function, the real part of which is the dispersion and the imaginary part is the absorption (§3.4.2). k_c is the plane wave phase $k_c = \omega_c/\lambda$. In a small range around ω_c, we consider the second-order approximation to be accurate enough, given the above slow-varying modulation condition (though, higher-order terms are frequently used in nonlinear optics). In hearing, the shifted frequency $\omega - \omega_c$ is referred to as the **envelope frequency** or **modulation frequency** and in communication theory the terms **frequency deviation** and **baseband frequency** are used.

We can plug Eq. 10.6 in Eq. 10.5 up to the second derivative of $k(\omega)$

$$\frac{\partial P(z,\omega)}{\partial z} = -i\left[k_c + \frac{dk}{d\omega}(\omega-\omega_c) + \frac{1}{2}\frac{d^2k}{d\omega^2}(\omega-\omega_c)^2\right]P(z,\omega) \tag{10.7}$$

If the spatial dependence of the envelope is shifted to be around ω_c, then $P(z,\omega)$ can be reformulated to factor out the constant phase using

$$P(z,\omega) = A(z,\omega-\omega_c)e^{-ik_cz} \tag{10.8}$$

Where the shifted complex spectral envelope around ω_c, $A(z,\omega-\omega_c)$, was introduced and now contains the spatial dependence in z through the higher order terms of k. Now Eq. 10.7 in P can be reformulated as an equation in A, eliminating the mean spatial frequency k_c

$$\frac{\partial A(z,\omega-\omega_c)}{\partial z} = -i\left[\frac{dk}{d\omega}(\omega-\omega_c) + \frac{1}{2}\frac{d^2k}{d\omega^2}(\omega-\omega_c)^2\right]A(z,\omega-\omega_c) \tag{10.9}$$

In order to obtain a time-domain expression of Eq. 10.9, it will be necessary to have the inverse Fourier transform of the shifted envelope, which can be obtained from the full signal and Eq. 10.8

$$\begin{aligned} p(z,t) &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} P(z,\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(z,\omega-\omega_c)e^{-ik_cz}e^{i\omega t}d\omega \\ &= \frac{1}{2\pi}e^{i\omega_ct-ik_cz}\int_{-\infty}^{\infty} A(z,\omega-\omega_c)e^{i(\omega-\omega_c)t}d(\omega-\omega_c) = a(z,t)e^{i\omega_ct-ik_cz} \end{aligned} \tag{10.10}$$

which means that the temporal envelope $a(z,t)$ is simply the inverse Fourier transform of the complex spectral envelope, in the modulation frequency coordinate

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A(z,\omega-\omega_c)e^{i(\omega-\omega_c)t}d(\omega-\omega_c) \equiv \mathcal{F}^{-1}\left[A(z,\omega-\omega_c)\right] = a(z,t) \tag{10.11}$$

This identity enables us to manipulate the complex envelope independently of the carrier, as a function of modulation frequency, in the narrowband approximation. Note this additional relation for the inverse Fourier transform

$$\mathcal{F}^{-1}\left\{[i(\omega-\omega_c)]^n A(z,\omega-\omega_c)\right\} = \frac{\partial^n}{\partial t^n}a(z,t) \tag{10.12}$$

The inverse Fourier transform can now be used to convert Eq. 10.9 to the time domain, using Eq. 10.12, which yields the following differential parabolic equation for the temporal pressure envelope

$$\left(\frac{\partial}{\partial z} + \frac{dk}{d\omega}\frac{\partial}{\partial t}\right)a(z,t) = i\left(\frac{1}{2}\frac{d^2k}{d\omega^2}\frac{\partial^2}{\partial t^2}\right)a(z,t) \tag{10.13}$$

In most physical conditions, frequency-dependent absorption is usually even with dependence on ω^2 (see § 3.4.2) and not on ω, so it is therefore assumed that $dk/d\omega$ is real. Then the (real) group velocity is defined as usual (Eq. 3.11)

$$\frac{1}{v_g} = \frac{dk}{d\omega} \tag{10.14}$$

Additionally, it is convenient to separate the real and imaginary parts of the second derivative of k. In general, $k(\omega) = \beta(\omega) + i\alpha(\omega)$, and the real part of its second derivative is

$$\beta'' = \mathrm{Re}\left(\frac{d^2k}{d\omega^2}\right) \tag{10.15}$$

which gives a measure of the **group-velocity dispersion** (GVD) of the medium, whereas the imaginary part relates to the absorption, or **gain dispersion** (Siegman, 1986, p. 335)

$$\alpha'' = \mathrm{Im}\left(\frac{d^2k}{d\omega^2}\right) \tag{10.16}$$

Finally, Eq. 10.13 may be further tidied up with change of variables to a traveling coordinate system, using

$$\tau = (t - t_0) - \frac{(z - z_0)}{v_g} \tag{10.17}$$

$$\zeta = z - z_0 \tag{10.18}$$

This change of variables means that the frequency dependence of a is always centered around the group velocity at ω_c, when it is situated at a distance ζ from the origin. The new time coordinate is in fact the difference between the phase time coordinate and the group delay of the pulse, measured against its reference at (t_0, z_0) (see §3.2 and Eq. 3.22). Using the chain rule on Eq. 10.13,

$$\frac{\partial a(\zeta,\tau)}{\partial \zeta}\frac{\partial \zeta}{\partial z} + \frac{\partial a(\zeta,\tau)}{\partial \tau}\frac{\partial \tau}{\partial z} + \frac{1}{v_g}\frac{\partial a(\zeta,\tau)}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{i\beta'' - \alpha''}{2}\left[\frac{\partial a^2(\zeta,\tau)}{\partial \tau^2}\left(\frac{\partial \tau}{\partial t}\right)^2 + \frac{\partial a(\zeta,\tau)}{\partial \tau}\left(\frac{\partial^2 \tau}{\partial t^2}\right)\right] \tag{10.19}$$

where we neglected the term with $\partial\zeta/\partial t = 0$. Then, after using the definitions of ζ and τ, the equation reduces to

$$\frac{\partial a(\zeta,\tau)}{\partial \zeta} = \frac{i\beta'' - \alpha''}{2}\frac{\partial^2 a(\zeta,\tau)}{\partial \tau^2} \tag{10.20}$$

As Eq. 10.20 incorporates complex dispersion, it combines effects from the diffusion equation (for absorption) and Schrödinger's equation (for dispersion). This coupling may significantly complicate the treatment of the imaging system, so in optical treatments of this equation the absorption term is generally neglected. As it turns out in this work, there is a broad range of useful results that can be obtained for hearing without resorting to absorption. However, a discussion about the significance of this term will be revisited in several places (mainly in §F). Therefore, in the remainder of this work, we will set $\alpha = 0$ and assume a "classical imaging system". Therefore,

$$\frac{\partial a(\zeta,\tau)}{\partial \zeta} = \frac{i\beta''}{2}\frac{\partial^2 a(\zeta,\tau)}{\partial \tau^2} \tag{10.21}$$

This is the parabolic dispersion equation, which belongs to the heat equation family and has a very similar mathematical form to the spatial paraxial equation (4.10). A solution can be given using an inverse Fourier transform on the initial spectrum at $z = 0$ (Haberman, 1983, pp. 349–354)[95], multiplied by a quadratic complex kernel

$$a(\zeta,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(0,\omega)\exp\left(-i\frac{\beta''\zeta}{2}\omega^2\right)e^{i\omega\tau}d\omega \tag{10.22}$$

[95] Note that the standard diffusion equation has interchanged time and space variables compared to the dispersion equation (10.21). As a result, the initial spectrum condition for the diffusion equation is at $t = 0$, whereas here it as at $\zeta = 0$.

Eq. 10.21 is the basic propagation transformation in a dispersive medium. Note the similarity to the Fresnel diffraction integral from spatial optics (Eq. 4.8). This solution, along with the paraxial-like differential equation 10.20, constitute for us the basic space-time analogy.

The term "paraxial" in geometrical optics refers to the sound rays that are considered in the analysis, which are in the vicinity of the optical axis. In wave optics, the same idea is expressed using the spatial frequency k, whose range expresses the limited angular variation around the optical axis. As the dispersion equation strictly deals with uniaxial plane waves, referring to this approximation as "paraxial" makes little sense here, and we shall instead rename Eq. 10.21 (and Eq. 10.20) to be the **Helmholtz paratonal equation**, in analogy to Eq. 4.10. Here, **paratonal** conveys the intention of the main approximation that we employ and is applicable to the auditory system—that of narrowband channels that are centered around a particular carrier, a characteristic frequency, or a tone. Arguably, the term "paratemporal" may be just as valid as a name, because temporal imaging is also based on the limited extent of the temporal aperture—a physical time window that effectively turns a continuous wave into a finite received pulse—in analogy to the finite extent of the spatial image §12.

It is important to dwell on the $\beta''\zeta/2$ factor in the exponent, of 10.22 which is what characterizes the group-velocity dispersion of the medium. Its units are $\mathrm{s}^2\,/\,\mathrm{rad}$, and the units of β'' are $s^2/\,\mathrm{rad\,m}$. As the group-velocity dispersion grows with increasing length of the dispersive path ζ, the total dispersion between the measurement points is of interest, rather than the specific magnitude of ζ. This may be avoided by using the group delay definition instead. For spectral phase dependnce around ω_c of the form

$$\varphi(\omega) = -k(\omega)\zeta \tag{10.23}$$

the group delay is (Eq. 3.21)

$$\tau_g = -\frac{d\varphi(\omega)}{d\omega} \tag{10.24}$$

so differentiating this definition produces an alternative expression for $\beta''z$ as well

$$\frac{d\tau_g}{d\omega} = -\frac{d^2\varphi}{d\omega^2} = \zeta\frac{d^2k}{d\omega^2} = \beta''\zeta \tag{10.25}$$

Therefore, the group-velocity dispersion parameter $\beta''z$ (that typically appears with the factor of $1/2$ from the Taylor expansion) expresses also the curvature of the frequency-dependent phase function. This equation also shows that the same information contained in the group-velocity dispersion is available in the group delay derivative. Therefore, we will always prefer to relate to group-delay dispersion (GDD) instead of group-velocity dispersion, because it better reflects the method of calculation, when the distance ζ is unknown.

Basic examples of explicit solutions to the paratonal equation and a review of useful related expressions are found in §B.

10.3 The time lens

The dispersion integral of Eq. 10.22 can be thought of as an all-pass filter in the frequency domain, which has a similar operation to the diffraction integral in the spatial frequency domain. Thus, it is curious to look for a propagation medium that can produce a multiplicative quadratic phase function that is analogous to the normal lens, but in the time coordinate instead of the spatial coordinate.

A **time lens** was defined in an analogous way to the spatial lens (Eq. 4.11), using the following phase function (Kolner and Nazarathy, 1989)

$$\varphi(\tau) = \frac{\omega_c\tau^2}{2f_T} \tag{10.26}$$

where the time-dependent phase (in the traveling pulse coordinate system) $\varphi(\tau)$ is a quadratic function of τ and is also dependent on the **focal time** f_T, the temporal equivalent to the focal length of the spatial lens. The phase function too can be generically expressed using Taylor series around τ_0

$$\varphi(\tau) = \varphi_0(\tau_0) + \frac{d\varphi}{d\tau}(\tau - \tau_0) + \frac{1}{2}\frac{d^2\varphi}{d\tau^2}(\tau - \tau_0)^2 + ... \tag{10.27}$$

Comparison with Eq. (10.26) at $\tau_0 = 0$ suggests that the focal time should be related to the phase with

$$f_T = \frac{\omega_c}{\frac{d^2\varphi}{d\tau^2}} \tag{10.28}$$

The respective response for a time lens with this phase function is

$$h_L(\tau) = \exp\left(i\frac{\omega_c\tau^2}{2f_T}\right) \tag{10.29}$$

This is the transfer function of a time-domain all-pass filter, which is mathematically analogous to the spatial lens. It will sometimes be more useful in its frequency-domain form, which can be obtained by Fourier transforming $h_L(\tau)$,

$$H_L(\omega) = \int_{-\infty}^{\infty} \exp\left(i\frac{\omega_c\tau^2}{2f_T}\right)e^{-i\omega\tau}d\tau = \sqrt{\frac{2\pi i f_T}{\omega_c}}\exp\left(-\frac{if_T\omega^2}{2\omega_c}\right) \tag{10.30}$$

Where $H_L(\omega)$ is the frequency-domain impulse response of the filter. This kind of integral repeats in several places along the text and is most easily solved using "Siegman's Lemma" (Siegman, 1986, p. 337), which is given by

$$\int_{-\infty}^{\infty} e^{-Bx^2-2Cx}dx = \sqrt{\frac{\pi}{B}}e^{C^2/B} \qquad \mathrm{Re}(B) > 0 \tag{10.31}$$

for any complex constants B and C. It can be readily derived by completing the square of the exponent.

Note that the focus f_T has the units of time, so the quantity $f_T/2\omega_c$ has the same units as the group dispersion factor $\beta''\zeta$ ($\mathrm{s}^2\,/\,\mathrm{rad}$). Thus, we define the lens curvature s,

$$s = \frac{f_T}{2\omega_c} \tag{10.32}$$

And Eq. 10.30 can be brought to the form (Kolner, 1994a)

$$H_L(\omega) = \sqrt{4\pi i s}\exp\left(-is\omega^2\right) \tag{10.33}$$

The time-lens processing redistributes the power of the various spectral components around the carrier—an operation that requires an active nonlinear device (Yariv and Yeh, 2007, pp. 278–279). In optics, such a lens is realized using a **phase modulator**, which is an active component unlike the passive dispersive medium. Kolner (1994a) emphasized that the ideal (electro-optic) phase modulator has to have a linear phase response—independent of the incoming wave amplitude. In general, the focal time is frequency-dependent, just as in spatial lenses that have a refraction index that depends on the wavelength of light and affects the lens curvature.

Different principles of phase modulation have been developed in optics, but the one that was favored by Kolner (1994a) is a traveling-wave modulator. It harnesses a microwave oscillation that is much lower frequency than the carrier and modulates the traversing wave phase as it passes through the device. Even if a reflected backward propagating wave inside the modulator exists, a good modulator should be coupled only to the forward-propagating mode. Optical modulators generally rely on the anisotropic index of refraction and two polarization modes of the electromagnetic radiation in the medium. Optical traveling-wave phase modulators work by slowly electro-optically modulating the index of refraction in a crystal, along which the light carrier propagates (Yariv and Yeh, 2007, pp. 429–431). The modulation frequency may be smaller than the carrier, or equal to it for maximum effect. However, analogous properties and related phenomena (e.g., the electro-optic and acousto-optic effects, which can slowly modulate light) do not exist in pure acoustic fields and other physical effects may have to be harnessed in order to obtain phase modulation. Therefore, more specific details about the optical phase modulators are not central to this analysis, as this is a point of divergence for the electromagnetic and acoustic scalar wave theories. It will be sufficient to know the mathematical principle, when we attempt to identify the relevant organ within the auditory system, even if other mechanisms may be conceived to realize these functions in acoustics. To the best knowledge of the author, phase modulators have not been systematically discussed outside of the photonics/optics literature.

10.4 Summary of assumptions

Throughout the above derivation, a number of assumptions were used beyond those of classical linear acoustics, which enabled the solution. The first three assumptions are synonymous with one another and can fall under the paratonal approximation definition:

1. Source: Narrowband signal
2. Source: Slow envelope in comparison with the carrier
3. Source/Medium: $k(\omega)$ changes slowly around the carrier; higher order terms than quadratic in the Taylor series are negligible
4. Medium: Source-free (secondary source wavefront)
5. Medium: Plane waves—one-dimensional propagation with no higher modes
6. Medium: Constant absorption around the carrier, but the derivatives of $k(\omega)$ are real, or much greater than the imaginary part
7. Time lens: phase function is level-independent (linear)
8. Time lens: phase function is quadratic

Unlike diffraction and scattering problems, non-planar spatial modes—functions that vary in the x and y dimensions—are neglected. Therefore, the paratonal equation is particularly attractive for communication, because limiting the carrier to a single (planar) mode eliminates any interaction between envelopes of different modes, which corrupts their shape. When light propagates in an optical fiber in different fiber modes that have different dispersions associated with them, the mode can interact (beat) and give rise to **dispersion distortion**[96]. In fiber optics, this is the prime reason for using single-mode fibers, whose dispersion is well-behaved, rather than multi-mode fibers that are prone to exhibit dispersion distortion, especially over long distances (Haus, 1984; Agrawal, 2001, pp. 1–17). Single-mode optical fibers are employed almost exclusively in the communication industry, where high channel capacity is required (Mitschke, 2009, p. 8). In the ideal design of single-mode optical fibers, the dispersive and absorptive effects of the fiber for particular carriers are minimal.

[96]This is **not** the same as intermodulation disortion, where different carrier frequencies interact. Dispersion distortion relates to the same frequencies carried in different modes with different group velocities associated with them. For a rare example in underwater acoustics, see Zhang et al. (2019).

10.5 Conclusion

The paratonal equation was introduced along with the simplest dispersive medium transformation and its active dual—the time lens. While these equations have not been used in acoustics or hearing before, they naturally fit them, given that real sources and their signals can be universally described as modulated carriers with complex envelopes. Not only do these equations provide a convenient mathematical analogy to the familiar spatial imaging theory from optics, but they also tackle the problem of envelope propagation, which has not received any rigorous closed-form treatment in acoustics.

At this point, all the basic components are available for constructing a complete temporal imaging system, based on a cascade of dispersion, time-lens, and another dispersion, in analogy to normal imaging with spatial lens that is sandwiched by diffraction. Before continuing to develop the theory for such a system in §12, we would first like to identify the different dispersive auditory elements and estimate their magnitudes in §11. This will ground the discussion about the complete imaging system to the relevant parameters of the ear.