Appendix D

Impulse response of rectangular pupil with positive defocus

The rectangular pupil can be informative as a limiting case, where extreme time-limitation of the object support is required. It can be solved in a closed form following the formulas provided in Klauder et al. (1960), but only for positive values of the defocus W_d. Let us define a rectangular pupil P_r of width T_a,

$$P_r(\tau) = \mathrm{rect}\left(\frac{\tau}{T_a}\right) = \begin{cases} 1 & |\tau| < T_a/2 \\ \frac{1}{2} & |\tau| = T_a/2 \\ 0 & |\tau| > T_a/2 \end{cases} \tag{D.1}$$

where T_a is equal here and in the Gaussian pupil, because of how we defined it in Eq. 13.19. Using this pupil in Eq. 13.18, we obtain the following integral

$$\tilde{h}_{dr}(\tau - \tilde{\tau}_0) = \frac{1}{2\pi} \exp\left(\frac{i\omega_c \tau^2}{2Mf_T}\right) \int_{-T_a/2}^{T_a/2} \exp(iv^2 W_d \tilde{T}^2) \exp\left[-i\tilde{T}(\tau - \tilde{\tau}_0)\right] d\tilde{T} \tag{D.2}$$

Then, the response would be given according to

$$\tilde{h}_{dr}(\tau - \tilde{\tau}_0) = \frac{1}{2\pi} \exp\left(\frac{i\omega_c \tau^2}{2Mf_T}\right) \sqrt{\frac{\pi}{2v^2 W_d}} \exp\left[-\frac{i(\tau - \tau_0)^2}{4\pi v^2 W_d}\right] \left[C(g_2) + iS(g_2) - C(g_1) - iS(g_1)\right] \tag{D.3}$$

Where C and S are the real and imaginary parts of the complex Fresnel integrals, respectively, which are defined as

$$S(g) = \int_0^g \sin(\psi^2) d\psi \tag{D.4}$$

$$C(g) = \int_0^g \cos(\psi^2) d\psi \tag{D.5}$$

The variables g_1 and g_2 are defined as

$$g_1(\tau) = \frac{\tau - \tilde{\tau}_0}{\sqrt{2\pi v^2 W_d}} + \sqrt{\frac{v^2 W_d T_a^2}{2\pi}} \tag{D.6}$$

$$g_2(\tau) = \frac{\tau - \tilde{\tau}_0}{\sqrt{2\pi v^2 W_d}} - \sqrt{\frac{v^2 W_d T_a^2}{2\pi}} \tag{D.7}$$

A numerical solution exists only when g is positive, which is unfortunately not the case in the auditory system using the values of v, T_a, and W_d obtained in this work.