# Appendix C

# Linear canonical transform approach to the dispersion integral

A general approach to a solution of the imaging transform in Eq. 12.13 is to reformulate the dispersion integral of the envelope as a **linear canonical transform** (LCT), similar to the Frensel-Kirchhoff diffraction integral in optics. In one-dimensional imaging over the time-frequency plane, this transform has the general kernel (e.g., Wolf, 1979, Chapter 9)

$$C_T(q, q') = \frac{1}{\sqrt{2\pi b}} e^{\frac{-i\pi}{4}} \exp\left[\frac{i}{2b}(aq^2 - 2q'q + dq'^2)\right] \qquad b \neq 0 \tag{C.1}$$

From this, the LCT of function $g(q)$ is

$$g_T(q') = \mathscr{L}(T)g(q) \equiv \int_{-\infty}^{\infty} g(q)C_T(q, q')dq \tag{C.2}$$

where the generic variables $q$ and $q'$ can be assigned the roles of either $f$ or $t$. Depending on the coefficients $a$, $b$, and $d$, the transform rotates the function $g(q)$ in the time-frequency phase plane (as can be obtained by the Wigner-Ville distribution, for example) at an arbitrary angle. Many of the LCT properties are identical to the Fourier transform, which is a special case when the phase rotation angle is $\pi/2$, or $-\pi/2$ for the inverse Fourier transform. The advantage of conforming to the LCT formulation is that it offers a sophisticated toolkit based on operator and matrix algebra, in which a cascade of operations on the input (transforms) can be realized and interpreted without having to solve the integrals explicitly (see for example, Shamir, 1999, Chapters 4, 12, and Appendix A). By comparing the coefficients of Eqs. (C.1) and (12.13), the parameters $a$, $b$, and $d$ can be found

$$a = 2\left(\frac{uv}{s} + v + u\right) \qquad b = -\frac{v+s}{s} \qquad d = -\frac{1}{2s} \tag{C.3}$$

assigning $q = \omega$ and $q' = \tau$. The transformation $T$ shall be defined as

$$T = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \tag{C.4}$$

And Eq. (12.13) can be rewritten as an LCT, only correcting for the amplitude by $e^{\frac{i\pi}{4}}/\sqrt{2\pi}$

$$a_n(\zeta_2, \tau) = \frac{e^{\frac{i\pi}{4}}}{\sqrt{2\pi}} \mathscr{L}(T)\left[A(0, \omega')\right] \tag{C.5}$$

Such a general transformation can always be decomposed into three cascaded operations (but other decompositions exist): a fractional Fourier transform that rotates the phase space, a magnifying operation, and a quadratic-phase modulation (Bastiaans and Alieva, 2016).